JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 4, November 2022
http://dx.doi.org/10.14403/jcms.2022.35.3.315

TRANSFERRED SUPERSTABILITY OF THE p-RADICAL SINE FUNCTIONAL EQUATION

Gwang Hui Kim* and Jaiok Roh**

Abstract. In this paper, we investigate the transferred superstability for the p-radical sine functional equation

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y)$$

from the p-radical functional equations:

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)g(y),$$
$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)h(y),$$

where p is an odd positive integer, λ is a positive real number, and f is a complex valued function. Furthermore, the results are extended to Banach algebras.

Therefore, the obtained result will be forced to the pre-results(p=1) for this type's equations, and will serve as a sample to apply it to the extension of the other known equations.

1. Introduction

The superstability of the cosine (d'Alembert) functional equation

$$f(x+y)+f(x-y)=2f(x)f(y) \tag{C}$$

was proved by Baker [4] in 1980.

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{W}$$
$$f(x+y)+f(x-y)=2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) arised by Kim was appeared in Kannappan and Kim's paper ([8]).

Received November 2, 2022; Accepted November 21, 2022.

2010 Mathematics Subject Classification: Primary 39B82, 39B52; Secondary 39B82.

Key words and phrases: stability, superstability, radical equation, sine functional equation, Wilson functional equation, Kim functional equation.

**Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C109489611).

The superstability of the cosine (C), Wilson (W) and Kim (K) functional equations were founded in Badora, Kannappan and Kim ([2, 8, 16]).

The superstability by constant bounded of the sine functional equation

$$\text{(S)} \qquad f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2,$$

is investigated by P.W. Cholewa [5], and is improved in R. Badora and R. Ger [3], and G. H. Kim [13], [15].

In 2009, Eshaghi Gordji and Parviz [7] introduced the radical functional equation

$$(R) \qquad f\big(\sqrt{x^2+y^2}\big) = f(x) + f(y)$$

related to the quadratic functional equation.

Recently, Almahalebi *et al.*[1] obtained the superstability in Hyer's sense for the p-radical functional equations related to Wilson equation and Kim's equation.

If the cosine functional equation (C) and the sine functional equation (S) is expressed in the concept of the p-radical function, respectively, that it is as following:

$$(C^r) \qquad f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)f(y),$$

$$(S^r) \qquad f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y).$$

Namely, applying $p = 1$ in (C^r) and (S^r), it implies (C) and (S).

The aim of this paper is to investigate the transferred superstability of the p-radical sine functional equation (S^r) related to the sine functional equation from the Pexider type p-radical functional equations

$$(K^r_{gg}) \qquad f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)g(y),$$

$$(K^r_{gh}) \qquad f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)h(y).$$

Furthermore, the results are extended to Banach spaces.

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f is a nonzero function, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function, and p is an odd nonnegative integer.

Let us denote the functional equations for the p-radical functional equations related to Wilson and Kim's equations as follows:

(C_λ^r) $\quad f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)f(y),$

(W^r) $\quad f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)g(y),$

(W_λ^r) $\quad f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y),$

(K^r) $\quad f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2g(x)f(y).$

(K_λ^r) $\quad f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y).$

(C_λ) $\quad f(x+y)+f(x-y)=\lambda f(x)f(y),$

(W_λ) $\quad f(x+y)+f(x-y)=\lambda f(x)g(y),$

(K_λ) $\quad f(x+y)+f(x-y)=\lambda g(x)f(y).$

2. Transferred superstability for the p-radical sine functional equation (S^r).

In this section, we investigate the transferred superstability for p-radical sine functional equation (S^r) from the p-radical functional equations (K_{gg}^r) and (K_{gh}^r).

In paper [9] , we can see a solution of the p-radical functional equations of the cosine(C^r), Wilson(W^r) and Kim(K^r) related to sine (S^r). In the following lemma, we obtain a solution of the p-radical sine functional equation (S^r).

Lemma 2.1. *A function $f:\mathbb{R}\to\mathbb{C}$ satisfies (S^r) if and only if $f(x)=\sin(x^p)=F(x^p)$ for all $x\in\mathbb{R}$, where F is a solution of* (S).

Proof. Namely,

$$F\left(\frac{x^p+y^p}{2}\right)^2-F\left(\frac{x^p-y^p}{2}\right)^2=\sin\left(\frac{x^p+y^p}{2}\right)^2-\sin\left(\frac{x^p-y^p}{2}\right)^2$$
$$=f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2-f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2=f(x)f(y)$$
$$=\sin\left(x^p\right)\sin\left(y^p\right)=F\left(x^p\right)F\left(y^p\right),\text{for all x,y}\in\mathbb{R}.$$

□

Theorem 2.2. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)g(y)|\le\varphi(x). \tag{2.1}$$

Then, either g (with $g(0)=0$ or $f(-x)=-f(x)$) is bounded or g satisfies (S^r).

Proof. (i) Assume that g is unbounded. Then we can choose $\{y_n\}$ such that $0 \neq |g(y_n)| \to \infty$ as $n \to \infty$.

Putting $y = y_n$ in (2.1) and dividing both sides by $\lambda g(y_n)$, we have

$$\left| \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda g(y_n)} - g(x) \right| \leq \frac{\varphi(x)}{\lambda g(y_n)}. \tag{2.2}$$

As $n \to \infty$ in (2.2), we get

$$g(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) + f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda g(y_n)} \tag{2.3}$$

for all $x \in \mathbb{R}$.

Replacing y by $\sqrt[p]{y^p + y_n^p}$ and $\sqrt[p]{y^p - y_n^p}$ in (2.1), we obtain

(2.4)
$$\left|\left(f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) - \lambda g(x) g(\sqrt[p]{y^p + y_n^p})\right| \leq \varphi(x),$$

(2.5)
$$\left|\left(f\left(\sqrt[p]{x^p + (y^p - y_n^p)}\right) + f\left(\sqrt[p]{x^p - (y^p - y_n^p)}\right) - \lambda g(x) g(\sqrt[p]{y^p - y_n^p})\right| \leq \varphi(x),$$

for all $x, y, y_n \in \mathbb{R}$. By (2.4) and (2.5), we obtain

$$\begin{aligned} &\Bigg| \frac{f\left(\sqrt[p]{(x^p + y^p) + y_n^p}\right) + f\left(\sqrt[p]{(x^p + y^p) - y_n^p}\right)}{\lambda g(y_n)} \\ &\quad + \frac{f\left(\sqrt[p]{(x^p - y^p) + y_n^p}\right) + f\left(\sqrt[p]{(x^p - y^p) - y_n^p}\right)}{\lambda g(y_n)} \\ &\quad - \lambda g(x) \frac{g(\sqrt[p]{y^p + y_n^p}) + g(\sqrt[p]{y^p - y_n^p})}{\lambda g(y_n)} \Bigg| \leq \frac{2\varphi(x)}{\lambda g(y_n)} \end{aligned} \tag{2.6}$$

for all $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$.

In inequality (2.6), taking the limit as $n \longrightarrow \infty$ with the using of (2.3), then, we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$L_1(y) := \lim_{n\to\infty} \frac{g(\sqrt[p]{y^p + y_n^p}) + g(\sqrt[p]{y^p - y_n^p})}{g(y_n)} \tag{2.7}$$

where the obtained function $L_1 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = g(x) L_1(y) \quad \forall x, y \in \mathbb{R}. \tag{2.8}$$

First, let us consider the case $g(0) = 0$, then it forces by (2.8) that g is odd. Putting $y = x$ in (2.8), we get

$$g(\sqrt[p]{2}x) = g(x) L_1(x), \quad \forall x \in \mathbb{R}. \tag{2.9}$$

From (2.8), the oddness of g and (2.9), we obtain the equation

$$\begin{aligned} g(\sqrt[p]{x^p+y^p})^2 - g(\sqrt[p]{x^p-y^p})^2 &= g(x)L_1(y)[g(\sqrt[p]{x^p+y^p}) - g(\sqrt[p]{x^p-y^p})] \\ &= g(x)[g(\sqrt[p]{x^p+2y^p}) - g(\sqrt[p]{x^p-2y^p})] \\ &= g(x)[g(\sqrt[p]{2y^p+x^p}) + g(\sqrt[p]{2y^p-x^p})] \\ &= g(x)g(\sqrt[p]{2}y)L_1(x) \\ &= g(\sqrt[p]{2}x)g(\sqrt[p]{2}y), \end{aligned} \tag{2.10}$$

that holds true for all $x, y \in \mathbb{R}$.

By putting $x = \frac{x}{\sqrt[p]{2}}, y = \frac{y}{\sqrt[p]{2}}$ in the inequality (2.10), then, it states nothing else but (S^r).

(ii) For next case $f(-x) = -f(x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case. Then, we may assume that $g(0) = c$: constant.

Putting $x = 0$ in (2.1), from the above assumption, we obtain the inequality

$$|g(y)| \leq \frac{\varphi(0)}{\lambda c} \qquad \forall\, y \in G.$$

This inequality means that g is globally bounded, which is a contradiction by unboundedness assumption. Thus the claimed $g(0) = 0$ holds, so the proof is completed. □

Theorem 2.3. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)g(y)| \leq \varphi(y) \tag{2.11}$$

Then, either g (with $g(0) = 0$) is bounded or g satisfies (S^r).

Proof. The proof follows from Theorem 2.2. Let us choose $\{x_n\}$ in $\mathbb{R}$ such that $0 \neq |g(x_n))| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ (with $n \in \mathbb{N}$) in (2.11), dividing both sides by $|\lambda g(x_n)|$, and passing to the limit as $n \to \infty$, we obtain that

$$g(y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p+y^p}\right) + f\left(\sqrt[p]{x_n^p-y^p}\right)}{\lambda g(x_n)} \tag{2.12}$$

for all $y \in \mathbb{R}$.

Replace (x, y) by $(\sqrt[p]{x_n^p+y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p-y^p}, x)$ in (2.11). Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of

Theorem 2.2. Then we obtain

$$
\begin{aligned}
&\Bigg|\frac{f\left(\sqrt[p]{(x_n^p+y^p)+x^p}\right)+f\left(\sqrt[p]{(x_n^p+y^p)-x^p}\right)}{\lambda g(x_n)}\\
&+\frac{f\left(\sqrt[p]{(x_n^p-y^p)+x^p}\right)+f\left(\sqrt[p]{(x_n^p-y^p)-x^p}\right)}{\lambda g(x_n)}\\
&-\lambda\frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{\lambda g(x_n)}g(x)\Bigg|\leq\frac{2\varphi(x)}{\lambda g(x_n)}.
\end{aligned}
\tag{2.13}
$$

In inequality (2.13), taking the limit as $n \longrightarrow \infty$ with the using of (2.12), then, we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$
L_2(y) := \lim_{n\to\infty}\frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{g(x_n)}
\tag{2.14}
$$

where the obtained function $L_2 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$
g(\sqrt[p]{x^p+y^p})+g(\sqrt[p]{x^p-y^p})=g(x)L_2(y)\quad\forall x,y\in\mathbb{R},
\tag{2.15}
$$

which is not other than (2.8).

The assumption $g(0) = 0$ in (2.15) forces that g is odd.

Hence, since the remainder of the proof is the same procedure as that (2.9) and (2.10) of Theorem 2.2, which completes the proof. □

The following corollaries follow immediately from Theorems 2.2 and 2.3.

Corollary 2.4. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$
\left|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)g(y)\right|\leq\varepsilon.
\tag{2.16}
$$

Then, either g (with $g(0) = 0$ or $f(-x) = -f(x)$) is bounded or g satisfies (S^r).

By a similar process of the proof of Theorems 2.2, 2.3, we can prove the following theorem.

Theorem 2.5. *Assume that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$
\left|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)h(y)\right|\leq\varphi(x).
\tag{2.17}
$$

Then, either h with $g(0) = 0$ is bounded or g satisfies (S^r),

Proof. Assume that h with $g(0) = 0$ is unbounded. By the same procedure as (2.2) $\sim$ (2.3) of Theorem 2.2, we get

$$
g(x)=\lim_{n\to\infty}\frac{f\left(\sqrt[p]{x^p+y_n^p}\right)+f\left(\sqrt[p]{x^p-y_n^p}\right)}{\lambda h(y_n)}
\tag{2.18}
$$

for all $x \in \mathbb{R}$.

Replacing y by $\sqrt[p]{y^p+y_n^p}$ and $\sqrt[p]{y^p-y_n^p}$ in (2.17).

Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of Theorem 2.2. Namely, for it, addition, dividing by $\lambda h(y_n)$,

This implies that

$$\begin{aligned}\Bigg| &\frac{f\left(\sqrt[p]{(x^p+y^p)+y_n^p}\right)+f\left(\sqrt[p]{(x^p+y^p)-y_n^p}\right)}{\lambda h(y_n)}\\ &+\frac{f\left(\sqrt[p]{(x^p-y^p)+y_n^p}\right)+f\left(\sqrt[p]{(x^p-y^p)-y_n^p}\right)}{\lambda h(y_n)}\\ &-\lambda g(x)\frac{h(\sqrt[p]{y^p+y_n^p})+h(\sqrt[p]{y^p-y_n^p})}{\lambda h(y_n)}\Bigg| \le \frac{2\varphi(x)}{\lambda h(y_n)}\end{aligned} \tag{2.19}$$

for all $x, y, y_n \in \mathbb{R}$.

In inequality (2.19), taking the limit as $n \longrightarrow \infty$ with the using of (2.18), then, we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$L_3(y) := \lim_{n\to\infty} \frac{h(\sqrt[p]{y^p+y_n^p})+h(\sqrt[p]{y^p-y_n^p})}{h(y_n)} \tag{2.20}$$

where the obtained function $L_3 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$g(\sqrt[p]{x^p+y^p})+g(\sqrt[p]{x^p-y^p}) = g(x)L_3(y) \quad \forall x, y \in \mathbb{R}. \tag{2.21}$$

From the assumption $g(0) = 0$, by (2.15) g is odd.

Thus, the remaining proof goes through the same procedure as after (2.9) and (2.10) in Theorem 2.2.

□

Theorem 2.6. *Assume that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)h(y)\right| \le \varphi(y). \tag{2.22}$$

Then, either g with $h(0) = 0$ is bounded or h satisfies (S^r),

Proof. Assume that g with $h(0) = 0$ is unbounded. By the same procedure as (2.2) $\sim$ (2.3) of Theorem 2.3, we deduce

$$h(y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p+y^p}\right)+f\left(\sqrt[p]{x_n^p-y^p}\right)}{\lambda g(x_n)} \tag{2.23}$$

for all $y \in \mathbb{R}$.

Replace (x, y) by $(\sqrt[p]{x_n^p+y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p-y^p}, x)$ in (2.23). Thereafter we go through the same procedure as in (2.4) $\sim$ (2.6) of

Theorem 2.2. Then we obtain

$$\begin{aligned}
&\Bigg|\frac{f\left(\sqrt[p]{(x_n^p+y^p)+x^p}\right)+f\left(\sqrt[p]{(x_n^p+y^p)-x^p}\right)}{\lambda g(x_n)} \\
&+\frac{f\left(\sqrt[p]{(x_n^p-y^p)+x^p}\right)+f\left(\sqrt[p]{(x_n^p-y^p)-x^p}\right)}{\lambda g(x_n)} \qquad (2.24)\\
&-\lambda\frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{\lambda g(x_n)}h(x)\Bigg| \le \frac{2\varphi(x)}{\lambda g(x_n)}.
\end{aligned}$$

Take the limit as $n \longrightarrow \infty$ with the use of $|g(x_n)| \to \infty$ in (2.24). Then, we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$L_4(y) := \frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{g(x_n)} \qquad (2.25)$$

where the obtained function $L_4 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$h(\sqrt[p]{x^p+y^p})+h(\sqrt[p]{x^p-y^p}) = h(x)L_4(y) \quad \forall x, y \in \mathbb{R}. \qquad (2.26)$$

In here, we can see that the obtained equation (2.26) is none other than (2.8). From the assumption $h(0) = 0$, by (2.15) h is odd.

Thus, the remaining proof goes through the same procedure as after (2.9) and (2.10) in Theorem 2.2. □

The following corollaries follow from Theorems 2.5 and 2.6.

Corollary 2.7. *[Theorem 1, [9]] Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)| \le \varphi(x) \qquad (2.27)$$

Then, either f is bounded or g satisfies (C_λ^r),

Proof. Applying h to f in (2.17), then (2.18) and (2.19) imply immediately that g satisfies (C_λ^r). □

Corollary 2.8. *[Theorem 1, [9]] Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)| \le \varphi(y) \text{ and } \varphi(x). \qquad (2.28)$$

Then, either g(or f) is bounded or g satisfies (C_λ^r), and f and g satisfy (K_λ^r) and (W_λ^r).

Proof. Let g be unbounded. We can show that f (or g) is unbounded if and only if g (or f) is also unbounded (see, Theorem 1,[9]). Hence, we can apply Corollary 2.7, it implies that g(or f) is bounded or g satisfies (C_λ^r).

Next, (i) replace x by $\sqrt[p]{x_n^p+x^p}, x)$ and $\sqrt[p]{x_n^p-x^p}$ in (2.28), respectively.

(ii) replace (x, y) by $(\sqrt[p]{x_n^p+y^p}, x)$ and replace (x, y) by $(\sqrt[p]{x_n^p-y^p}, x)$ in (2.28), respectively.

Then, applying the above result, due to replaced h by f in (2.25) to (2.26), the remainder also arrives smoothly. □

Corollary 2.9. *[Theorem 2, [9]] Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)| \le \begin{cases} (i) \ \ \varphi(y) \\ (ii) \ \ \varphi(x) \ \ \text{and} \ \ \varphi(y). \end{cases}$$

Then

(i) either f is bounded or g satisfies (C_λ^r),

(ii) either g(or f) is bounded or g satisfies (C_λ^r), and f and g satisfy (K_λ^r) and (W_λ^r).

Proof. Replacing g by f and h by g in Theorems (2.5) and (2.6), then it is completed by the same process as Corollaries (2.8) and (2.9). □

Corollary 2.10. *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)f(y)| \le \begin{cases} (i) \ \varphi(x), \\ (ii) \ \varphi(y), \\ (iii) \ \varepsilon. \end{cases}$$

Then either f is bounded or f satisfies (C_λ^r),

Proof. Replacing g by f in Corollaries (2.8) or (2.9). □

Remark 2.11. In all results, applying $p = 1$ or $\lambda = 2$, one can obtain (C), (W), (K), (C_λ), (W_λ), (K_λ) (C^r), (W^r), (K^r). Hence they can be applied to stability results of the cosine, Wilson, Kim, trigonometric functional eqautions, etc. See Badora [2], Badora and Ger [3], Baker [4], Fassi, et al.[6], Kannappan and Kim [8], Kim [11, 16], and Almahalebi, et al.[1].

3. Extension to Banach algebras

In this section, all results in Section 2 will be extended to Banach algebras. Since the same applies to all results, the main theorems 2.5 and 2.6 are only grouped together and the rest of the results will be omitted.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)h(y)\| \le \begin{cases} (i) \ \ \varphi(x) \\ (ii) \ \ \varphi(y). \end{cases} \tag{3.1}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ h$ with $g(0) = 0$ is unbounded, then g satisfies (S^r).

(ii) If $z^* \circ g$ with $h(0) = 0$ is unbounded, then h satisfies (S^r).

Proof. Assume that (3.5) holds and let $z^* \in E^*$ be a linear multiplicative functional. Since $\|z^*\| = 1$, for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\varphi(x) &\geq \left\| f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)h(y)\right\| \\
&= \sup_{\|w^*\|=1} \left| w^*\big(f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)h(y)\big)\right| \\
&\geq \left| z^*\big(f\left(\sqrt[p]{x^p + y^p}\right)\big) + z^*\big(f\left(\sqrt[p]{x^p - y^p}\right)\big) - \lambda \cdot z^*\big(g(x)\big) \cdot z^*\big(h(y)\big)\right|,
\end{aligned}$$

which states that the superpositions $z^* \circ g$ and $z^* \circ h$ yield solutions of the inequalities ((2.17) and (2.22) in Theorems 2.5 and 2.6, respectively.

(i) Hence we can apply to Theorem 2.5.

Since, by assumption, the superposition $z^* \circ h$ with $g(0) = 0$ is unbounded, an appeal to Theorem 2.5 shows that the superposition $z^* \circ g$ is a solution of (S^r), that is,

$$(z^* \circ g)\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - (z^* \circ g)\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 = \lambda(z^* \circ g)(x)(z^* \circ g)(y).$$

Since z^* is a linear multiplicative functional, we get

$$z^*\left(g\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)g(y)\right) = 0.$$

Hence an unrestricted choice of z^* implies that

$$g\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)g(y) \in \bigcap\{\ker z^* : z^* \in E^*\}.$$

Since E is a semisimple Banach algebra, $\bigcap\{\ker z^* : z^* \in E^*\} = 0$, which means that g satisfies the claimed equation (S^r).

(ii) By assumption, the superposition $z^* \circ g$ with $h(0) = 0$ is unbounded, an appeal to Theorem 2.6 shows that the results hold.

An appeal to Theorem 2.6 shows that $z^* \circ h$ is solution of the equation (S^r), that is,

$$(z^* \circ h)\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - (z^* \circ h)\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 = \lambda(z^* \circ h)(x)(z^* \circ h)(y).$$

This means by a linear multiplicativity of z^* that the differences

$$\mathcal{D}S^r(x, y) := h\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - h\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - h(x)h(y)$$

fall into the kernel of z^*. That is, $z^*\left(\mathcal{D}S^r(x, y)\right) = 0$.

Hence an unrestricted choice of z^* implies that

$$\mathcal{D}S^r(x, y) \in \bigcap\{\ker z^* : z^* \in E^*\}.$$

Since the algebra E is semisimple, $\bigcap\{\ker z^* : z^* \in E^*\} = 0$, which means that h satisfies the claimed equations (S^r). $\square$

By a similar procedure, we can prove the next theorem as an extension of Theorems 2.2 and 2.3. So we will skip the proof.

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)g(y)\| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y). \end{cases} \tag{3.2}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ g$ with $g(0) = 0$ or $f(-x) = -f(x)$ is unbounded, then h satisfies (S^r).

(ii) If $z^* \circ g$ with $g(0) = 0$ is unbounded, then g satisfies (S^r).

Corollary 3.3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)f(y)\| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \text{ and } \varphi(x). \end{cases}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (C_λ^r).

(ii) If $z^* \circ g$ (or $z^* \circ f$) is unbounded, then g satisfies (C_λ^r), and f and g satisfy (K_λ^r) and (W_λ^r).

Corollary 3.4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)\| \leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \text{ and } \varphi(y). \end{cases} .$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional.

(i) If $z^* \circ f$ is unbounded, then g satisfies (C_λ^r).

(ii) If $z^* \circ g$ (or $z^* \circ f$) is unbounded, then g satisfies (C_λ^r), and f and g satisfy (K_λ^r) and (W_λ^r).

Corollary 3.5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)f(y)\| \leq \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \\ (iii) & \varepsilon. \end{cases}$$

Then either the superposition $z^* \circ f$ is bounded for each linear multiplicative functional $z^* \in E^*$ or f satisfies (C_λ^r).

REMARK 3.6. As like 2.11, in all results, applying $p = 1$ or $\lambda = 2$, one can obtain (C), (W), (K), (C_λ), (W_λ), (K_λ) (C^r), (W^r), (K^r). Hence they can be obtained to the stability results on the Banach algebras of the cosine, Wilson, Kim, trigonometric functional eqautions, etc. See Badora [2], Badora and Ger [3], Baker [4], Fassi, et al.[6], Kannappan and Kim [8], Kim [11, 16], and Almahalebi, et al.[1].

Acknowledgment

This paper was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C109489611).

References

[1] M. Almahalebi, R. El Ghali, S. Kabbaj and C. Park, *Superstability of p-radical functional equations related to Wilson–Kannappan–Kim functional equations*, Results Math., **76** (2021), 97-105 https://doi.org/10.1007/s00025-021-01409-2

[2] R. Badora, *On the stability of cosine functional equation*, Rocznik Naukowo-Dydak. Prace Mat., **15** (1998), 1-14.

[3] R. Badora and R. Ger, *On some trigonometric functional inequalities in Functional Equations*, Results and Advances, (2002), 3-15.

[4] J. A. Baker, *The stability of the cosine equation*, Proc. Am. Math. Soc., **80** (1980), 411-416.

[5] P. W. Cholewa, *The stability of the sine equation*, Proc. Am. Math. Soc., **88** (1983), 631-634.

[6] Iz. EL-Fassi, S. Kabbaj and G. H. Kim, *Superstability of a Pexider-type trigonometric functional equation in normed algebras*, Inter. J. Math. Anal., **9** (2015), no. 58, 2839-2848.

[7] M. Eshaghi Gordji and M. Parviz, *On the Hyers-Ulam-Rassias stability of the functional equation* $f(\sqrt{x^2+y^2}) = f(x) + f(y)$, Nonlinear Funct. Anal. Appl., **14** (2009), 413-420.

[8] Pl. Kannappan and G. H. Kim, *On the stability of the generalized cosine functional equations*, Ann. Acad. Pedagog. Crac. Stud. Math., **1** (2001), 49-58.

[9] G. H. Kim, *Superstability of the p-radical functional equations related to Wilson equation and Kim's eqaution*, Int. J. Nonlinear Anal. Appl., **12** (2021), 571-582.

[10] G. H. Kim, *Superstability and Solution of The Pexiderized Trigonometric Functional Equation*, Mathematics and Statistics, **8** (2020), no. 3, 363-371. DOI: 10.13189/ms.2020.080316

[11] G. H. Kim, *Superstability of some Pexider-type functional equation*, J. Inequal. Appl., **2010** (2010), DOI:10.1155/2010/985348.

[12] G. H. Kim and Y. H. Lee, *Boundedness of approximate trigonometric functional equations*, Appl. Math. Lett., **331** (2009), 439-443.

[13] G. H. Kim, *On the stability of the generalized sine functional equations*, Acta Math. Sinica, **25** (2009), 965-972.

[14] G. H. Kim, *On the stability of the Pexiderized trigonometric functional equation*, Appl. Math. Comput., **203** (2008), 99-105.

[15] G. H. Kim, *A Stability of the generalized sine functional equations*, Jour. Math. Anal & Appl., **331** (2007), 886-894.

[16] G. H. Kim, *The stability of the d'Alembert and Jensen type functional equations*, J. Math. Anal. Appl., **325** (2007), 237-248.

[17] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc., **72** (1978), 297-300.

Gwang Hui Kim
Department of Mathematics
Kangnam University
Yongin, Gyeonggi, 16979, Republic of Korea
E-mail: ghkim@kangnam.ac.kr

Jaiok Roh
Ilsong Liberal Art Schools(Mathematics)
Hallym University
Chuncheon, Kangwon-Do 200-702, Republic of Korea
E-mail: joroh@hallym.ac.kr